NASA TECHNICAL NOTE

NASA TN D-6939

NASA TN D-6939

EMISSION OF SOUND FROM AXISYMMETRIC TURBULENCE CONVECTED BY A MEAN FLOW WITH APPLICATION TO JET NOISE

by Marvin E. Goldstein and Burt M. Rosenbaum

Lewis Research Center
Cleveland, Ohio 44135

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • SEPTEMBER 1972

1. Report No. NASA TN D-6939	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle EMISSION OF SOUND FROM AXISYMMETRIC TURBULENCE CONVECTED BY A MEAN FLOW WITH APPLICATION TO JET NOISE		5. Report Date September 1972
		6. Performing Organization Code
7. Author(s) Marvin E. Goldstein and Burt M. Rosenbaum		8. Performing Organization Report No. E-6844
		10. Work Unit No. 113-31
9. Performing Organization Name and Address Lewis Research Center National Aeronautics and Space Administration Cleveland, Ohio 44135		11. Contract or Grant No.
		13. Type of Report and Period Covered Technical Note
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

A model, based on Lighthill's theory, for predicting aerodynamic noise from a turbulent shear flow is developed. This model is a generalization of the one developed by Ribner. Unlike Ribner's model, it does not require that the turbulent correlations factor into space- and time-dependent parts. It replaces his assumption of isotropic turbulence by the more realistic one of axisymmetric turbulence. The implications of the model for jet noise are discussed.

17. Key Words (Suggested by Author(s)) Acoustics Aerodynamic sound Jet noise	18. Distribution Statement Unclassified - unlimited

19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 41	22. Price* $3.00

* For sale by the National Technical Information Service, Springfield, Virginia 22151

CONTENTS

EMISSION OF SOUND FROM AXISYMMETRIC TURBULENCE CONVECTED BY A MEAN FLOW WITH APPLICATION TO JET NOISE

by Marvin E. Goldstein and Burt M. Rosenbaum

Lewis Research Center

SUMMARY

A model, based on Lighthill's theory, for predicting aerodynamic noise from a turbulent shear flow is developed. This model is a generalization of the one developed by Ribner. Unlike Ribner's model, it does not require that the turbulent correlations factor into space- and time-dependent parts. It replaces his assumption of isotropic turbulence by the more realistic one of axisymmetric turbulence. The implications of the model for jet noise are discussed.

INTRODUCTION

In order to use Lighthill's equation (ref. 1) to predict the sound intensity from the known properties of a turbulent shear flow, it is necessary to measure or deduce analytically the two-point space-time correlations of the second time derivatives of the Reynolds stresses and then integrate the results over the turbulent region. An evaluation of this kind either by exact analytical methods or experimentally in three dimensions would be extremely difficult if not impossible. In fact, the only case where the sound emission from turbulent flow has been explicitly calculated is for decaying isotropic turbulence (ref. 2) which is not directly applicable to jet noise. It is therefore necessary to develop approximate models of the turbulence which will allow the noise to be predicted in terms of a fairly small number of (possibly experimentally determined) quantities. There have been models of this type developed by both Lilley (ref. 3) and Ribner (ref. 4). However, both of these treatments are based on the assumption of locally homogeneous and isotropic turbulence and Ribner's treatment on the assumption that the space-time correlations of the Reynolds stresses factor into the product of a function of space and a function of time.

In the present report we shall carry through the analysis without making the latter assumption. The experimental results of reference 5 show that the assumption of isotropic turbulence may not be a good approximation for the correlations of the Reynolds stresses needed to predict jet noise. A much better assumption would be that the turbulence is axisymmetric about the direction of flow. The theory of axisymmetric turbulence was developed by Batchelor and Chandrasekhar (refs. 6 and 7). In axisymmetric turbulence, the statistical properties of the turbulence are the same in every direction transverse to the flow. But the properties in the direction of flow can be different from those properties in the directions transverse to the flow. The model obtained is more complicated than Ribner's but reduces to his in the case of isotropic turbulence. This reduction occurs without making the assumption that the correlation factor into a space- and time-dependent part. This shows that Ribner's model is actually more general than his assumptions indicate. It is shown in the course of the analysis that the decomposition of the sound intensity into shear and self-noise with no coupling terms occurs under the relatively weak assumption that the turbulence is locally homogeneous. (Note that isotropic turbulence must also be homogeneous (ref. 8, p. 3).) In fact, a hierarchy of equations obtained by making progressively more restrictive assumptions about the turbulence is presented in tabular form. Certain implications of the model for jet noise are discussed. It is shown that this model contains both the contributions to the shear noise term postulated by Lighthill and those postulated by Ribner, thus to some degree unifying these two approaches.

BASIC EQUATIONS

Lighthill (ref. 1) has shown that at a point $\vec{x}$ far enough from the flow to be in the radiation field of each turbulent eddy (that is, at a distance large compared with $(2\pi)^{-1}$ times a typical wavelength) the density fluctuations radiated by a localized turbulent flow are given by

$$\rho - \rho_o \sim \frac{1}{4\pi c_o^2} \int \frac{(x_i - y_i)(x_j - y_j)}{|\vec{x} - \vec{y}|^3} \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \tau_{ij}\left(\vec{y}, t - \frac{|\vec{x} - \vec{y}|}{c_o}\right) d\vec{y} \tag{1}$$

where c_o and ρ_o are, respectively, the speed of sound and mean density outside the jet and the integration is carried out over the region of turbulence. The source function τ_{ij} is given approximately by

$$\tau_{ij} \approx \rho_o v_i v_j \tag{2}$$

where $v_i(\vec{y})$ is the i^{th} component of the total velocity at the point $\vec{y}$. Since we are dealing with a stationary process in time, we shall suppose (as usual in order to insure convergence of the Fourier integrals) that $\tau_{ij} = 0$ for $|t| > T$ where T is some large time which will be put equal to infinity at the end of the analysis. We now introduce the Fourier transforms

$$\Delta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} (\rho - \rho_o) dt$$

$$T_{ij} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \tau_{ij} \, dt \tag{3}$$

Then equation (1) becomes

$$\Delta = \frac{-\omega^2}{4\pi c_o^4} \int \frac{(x_i - y_i)(x_j - y_j)}{|\vec{x} - \vec{y}|^3} \exp\left[i\omega \frac{|\vec{x} - \vec{y}|}{c_o} \right] T_{ij} \, dy \tag{4}$$

At any point in the far field, the spectral density of the intensity $I(\vec{x})$ is

$$I_\omega(\vec{x}) = \frac{c_o^3}{\rho_o} \frac{|\Delta|^2}{2T}$$

Substituting equation (4) in this expression yields

$$I_\omega(\vec{x}) = \frac{\omega^4}{16\pi^2 \rho_o c_o^5} \iint \frac{(x_i - y_i')(x_j - y_j')(x_k - y_k'')(x_l - y_l'')}{|\vec{x} - \vec{y}'|^3 |\vec{x} - \vec{y}''|^3} \exp\left\{ \frac{i\omega}{c_o} \left[|\vec{x} - \vec{y}'| - |\vec{x} - \vec{y}''| \right] \right\} \frac{T_{ij}(\vec{y}')T_{kl}^*(\vec{y}'')}{2T} \, d\vec{y}' \, d\vec{y}'' \tag{5}$$

Now upon using the fact that the Fourier transform of a convolution is proportional to the product of the Fourier transforms of its components and the fact that the Fourier transform of $f(-t)$ is the complex conjugate of the Fourier transform of $f(t)$ provided $f(t)$ is real, it is easy to see from equations (2) and (3) that

$$\frac{T_{ij}(y')T_{kl}^*(y'')}{2T} = \frac{\rho_o^2}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau} \overline{v_i' v_j' v_k'' v_l''} \, d\tau \tag{6}$$

where

$$\overline{v_i'v_j'v_k''v_l''} \equiv \frac{1}{\rho_0^2 2T} \int_{-T}^{T} \tau_{ij}(\vec{y}',t)\tau_{kl}(\vec{y}'',t+\tau)dt = \frac{1}{2T}\int_{-T}^{T} v_i(\vec{y}',t)v_j(\vec{y}',t)v_k(\vec{y}'',t+\tau)v_l(\vec{y}'',t+\tau)dt$$

is a fourth-order two-point two-time velocity correlation tensor.

TURBULENCE CORRELATION FUNCTIONS

We suppose that the mean velocity is in the y_1-direction and changes only in the y_2-direction.[1] Then

$$v_i(\vec{y},t) = \delta_{i1}U(y_2) + u_i(\vec{y},t)$$

where U is the mean velocity and u_i is the fluctuating (turbulent) velocity. Upon introducing this into equation (7), we find that

$$\overline{v_i'v_j'v_k''v_l''} = \mathscr{R}_{ijkl}^{(0)}(\vec{y}',\vec{\eta}) + \frac{1}{2}\mathscr{R}_{ijkl}^{(1)}(\vec{y}',\vec{\eta},\tau) + \frac{1}{2}\mathscr{R}_{ijkl}^{(2)}(\vec{y}'',\vec{\eta}^{(1)},\tau) \tag{8}$$

where we have put

$$\vec{\eta} = -\vec{\eta}^{(1)} = \vec{y}'' - \vec{y}' \tag{9}$$

$$\mathscr{R}_{ijkl}^{(0)}(\vec{y}',\vec{\eta}) \equiv U'^2\delta_{1i}\delta_{1j}\overline{u_k''u_l''} + U''^2\delta_{1k}\delta_{1l}\overline{u_i'u_j'} + U'^2U''^2\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} + \overline{u_i'u_j'}\;\overline{u_k''u_l''}$$

$$\begin{aligned}\mathscr{R}_{ijkl}^{(1)}(\vec{y}',\vec{\eta},\tau) \equiv {} & \overline{u_i'u_j'u_k''u_l''} - \overline{u_i'u_j'}\;\overline{u_k''u_l''} + 2U'\left(\delta_{1i}\overline{u_j'u_k''u_l''} + \delta_{1j}\overline{u_i'u_k''u_l''}\right) \\ & + U'U''\left(\delta_{1i}\delta_{1k}\overline{u_j'u_l''} + \delta_{1j}\delta_{1l}\overline{u_i'u_k''} + 2\delta_{1j}\delta_{1k}\overline{u_i'u_l''}\right)\end{aligned} \tag{10}$$

---

[1]This is a reasonable assumption for a jet if we suppose that $\vec{y}$ is a locally Cartesian coordinate system at each point of the mixing region.

$$\mathscr{R}^{(2)}_{ijkl}(\vec{y}'', \vec{\eta}^{(1)}, \tau) \equiv \overline{u_i' u_j' u_k'' u_l''} - \overline{u_i' u_j'}\;\overline{u_k'' u_l''} + 2U''\left(\delta_{1k}\overline{u_i' u_j' u_l''} + \delta_{1l}\overline{u_i' u_j' u_k''}\right)$$

$$+ U'U''\left(\delta_{1i}\delta_{1k}\overline{u_j' u_l''} + \delta_{1j}\delta_{1l}\overline{u_i' u_k''} + 2\delta_{1i}\delta_{1l}\overline{u_j' u_k''}\right) \qquad (11)$$

and the double primes indicate that the quantities are to be evaluated at $\vec{y}'' = \vec{y}' + \vec{\eta}$ and $t + \tau$ while the primed quantities are evaluated at $\vec{y}'$ and t. Notice that $\mathscr{R}^{(0)}_{ijkl}(\vec{y}', \vec{\eta})$ is independent of τ. Hence

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\mathscr{R}^{(0)}_{ijkl}(\vec{y}', \vec{\eta})d\tau = \delta(\omega)\mathscr{R}^{(0)}_{ijkl}$$

where $\delta(\omega)$ is the delta function. Since $\omega\delta(\omega) = 0$, we see that, when equation (8) is substituted into equation (6) and the result substituted into equation (5), the term $\mathscr{R}^{(0)}_{ijkl}(\vec{y}', \vec{\eta})$ makes no contribution to the integral. Hence, upon making this substitution and changing the variables of integration from $\vec{y}'$ and $\vec{y}''$ to $\vec{y}'$ and $\vec{\eta}$ for the integration over $\mathscr{R}^{(1)}_{ijkl}$ and from $\vec{y}'$ and $\vec{y}''$ to $\vec{y}''$ and $\vec{\eta}^{(1)}$ for the integration over $\mathscr{R}^{(2)}_{ijkl}$, we find that

$$I_\omega(x) = \frac{1}{2}\frac{\omega^4\rho_o}{32\pi^3c_o^5}\iiint \frac{(x_i - y_i')(x_j - y_j')(x_k - y_k' - \eta_k)(x_l - y_l' - \eta_l)}{|\vec{x} - \vec{y}'|^3|\vec{x} - \vec{y}' - \vec{\eta}|^3}\mathscr{R}^{(1)}_{ijkl}(\vec{y}', \vec{\eta}, \tau)\exp\left\{i\omega\left[\frac{1}{c_o}(|\vec{x} - \vec{y}'|) - |\vec{x} - \vec{y}' - \vec{\eta}|) - \tau\right]\right\}d\vec{\tau}\,d\vec{y}'\,d\vec{\eta}$$

$$+ \frac{1}{2}\frac{\omega^4\rho_o}{32\pi^3c_o^5}\iiint \frac{\left(x_i - y_i'' - \eta_i^{(1)}\right)\left(x_j - y_j'' - \eta_j^{(1)}\right)(x_k - y_k'')(x_l - y_l'')}{|\vec{x} - \vec{y}'' - \vec{\eta}^{(1)}|^3|\vec{x} - \vec{y}''|^3}\mathscr{R}^{(2)}_{ijkl}(\vec{y}'', \vec{\eta}^{(1)}, \tau)\exp\left\{i\omega\left[\frac{1}{c_o}(|\vec{x} - \vec{y}'' - \vec{\eta}^{(1)}| - |\vec{x} - \vec{y}''|) - \tau\right]\right\}d\vec{\tau}\,d\vec{y}''\,d\vec{\eta}^{(1)}$$

But since the variables $\vec{y}''$ and $\vec{\eta}^{(1)}$ in the second integral are variables of integration, we can replace them by $\vec{y}'$ and $\vec{\eta}$, respectively. Also since i, j, k, and l are dummy indices, we can interchange i with k and j with l without changing the value of this integral. Thus, the second integral in equation (12) becomes, after changing the variable τ to $-\tau$,

$$\frac{1}{2}\frac{\omega^4\rho_o}{32\pi^3c_o^5}\iiint \frac{(x_i - y_i')(x_j - y_j')(x_k - y_k' - \eta_k)(x_l - y_l' - \eta_l)}{|\vec{x} - \vec{y}'|^3|\vec{x} - \vec{y}' - \vec{\eta}|^3}\mathscr{R}^{(2)}_{klij}(\vec{y}', \vec{\eta}, -\tau)\exp\left\{-i\omega\left[\frac{1}{c_o}(|\vec{x} - \vec{y}'| - |\vec{x} - \vec{y}' - \vec{\eta}|) - \tau\right]\right\}d\tau\,d\vec{y}'\,d\vec{\eta}$$

Because the turbulence is assumed to be a stationary process in time, the correlations must be invariant under a translation in time. Consequently, we can replace t in $\mathscr{R}^{(2)}_{klij}(\vec{y}', \vec{\eta}, -\tau)$ by $t + \tau$. It is now easy to see that replacing $\vec{y}''$ by $\vec{y}'$, $\vec{\eta}^{(1)}$ by $\vec{\eta}$ and

τ by $-\tau$ in equation (11) merely interchanges the primed and double-primed quantities. Hence we see from equations (10) and (11) that

$$\mathscr{R}^{(1)}_{ijkl}(\vec{y}',\vec{\eta},\tau) = \mathscr{R}^{(2)}_{klij}(\vec{y}',\vec{\eta},-\tau)$$

Upon inserting this relation into equation (13), we find that the second term on the right-hand side of equation (12) is merely the complex conjugate of the first. Therefore,

$$I_\omega(\vec{x}) = \frac{\omega^4 \rho_o}{32\pi^3 c_o^5} \mathcal{Re} \iiint \frac{(x_i - y_i')(x_j - y_j')(x_k - y_k' - \eta_k)(x_l - y_l' - \eta_l)}{|\vec{x} - \vec{y}'|^3 |\vec{x} - \vec{y}' - \vec{\eta}|^3} \mathscr{R}^{(1)}_{ijkl}(\vec{y}',\vec{\eta},\tau) \exp\left\{i\omega\left[\frac{1}{c_o}(|\vec{x} - \vec{y}'| - |\vec{x} - \vec{y}' - \vec{\eta}|) - \tau\right]\right\} d\tau\, d\vec{y}'\, d\vec{\eta}$$

EXPANSION FOR OBSERVATION POINT FAR FROM FLOW

We first introduce the new variable

$$\vec{y} \equiv \left(y_1', \frac{y_2' + y_2''}{2}, \frac{y_3' + y_3''}{2}\right) = \frac{\vec{y}' + \vec{y}''}{2} + \hat{i}\left(\frac{y_1' - y_1''}{2}\right) \tag{15}$$

where $\hat{i}$ denotes the unit vector in the y_1-direction. Then

$$\vec{y}' = \vec{y} + \frac{1}{2}(\hat{i}\,\eta_1 - \vec{\eta})$$

Hence, if in equation (14) we put

$$\mathscr{R}^{(1)}_{ijkl}(\vec{y}',\vec{\eta},\tau) = \mathscr{R}^{+}_{ijkl}(\vec{y},\vec{\eta},\tau) + \mathscr{R}^{c}_{ijkl}(\vec{y}',\vec{\eta},\tau)$$

where

$$\mathscr{R}^{+}_{ijkl}(\vec{y},\vec{\eta},\tau) \equiv \overline{u_i'u_j'u_k''u_l''} - \overline{u_i'u_j'}\;\overline{u_k''u_l''} + U'U''(\delta_{1i}\delta_{1k}\overline{u_j'u_l''} + \delta_{1j}\delta_{1l}\overline{u_i'u_k''} + 2\delta_{1j}\delta_{1k}\overline{u_i'u_l''}) \tag{16}$$

$$\mathscr{R}^{c}_{ijkl}(\vec{y}',\eta,\tau) \equiv 2U'(\delta_{1i}\overline{u_j'u_k''u_l''} + \delta_{1j}\overline{u_i'u_k''u_l''})$$

and then change the variables of integration from $\vec{y}'$ and $\vec{\eta}$ to $\vec{y}$ and $\vec{\eta}$ in performing the integration over $\mathscr{R}^{+}_{ijkl}$, we obtain

$$I_\omega(\vec{x}) = \frac{\omega^4 \rho_o}{32\pi^3 c_o^5} \mathcal{Re}\Bigg\{\iiint \frac{\left(\zeta_i - y_i + \frac{1}{2}\eta_i\right)\left(\zeta_j - y_j + \frac{1}{2}\eta_j\right)\left(\zeta_k - y_k - \frac{1}{2}\eta_k\right)\left(\zeta_l - y_l - \frac{1}{2}\eta_l\right)}{|\vec{\zeta} - \vec{y} + \frac{1}{2}\vec{\eta}|\,|\vec{\zeta} - \vec{y} - \frac{1}{2}\vec{\eta}|} \exp\left[\frac{i\omega}{c_o}\left(|\vec{\zeta} - \vec{y} + \frac{1}{2}\vec{\eta}| - |\vec{\zeta} - \vec{y} - \frac{1}{2}\vec{\eta}| - c_o\tau\right)\right] \mathscr{R}^{+}_{ijkl}(\vec{y},\vec{\eta},\tau)\, d\tau\, d\vec{y}\, d\vec{\eta}$$

$$+ \iiint \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k - \eta_k)(x_l - y_l - \eta_l)}{|\vec{x} - \vec{y}|^3 |\vec{x} - \vec{y} - \vec{\eta}|^3} \mathscr{R}^{c}_{ijkl}(\vec{y},\vec{\eta},\tau) \exp\left[\frac{i\omega}{c_o}(|\vec{x} - \vec{y}| - |\vec{x} - \vec{y} - \vec{\eta}| - c_o\tau)\right] d\tau\, d\vec{y}\, d\vec{\eta}\Bigg\}$$

where we have put for brevity

$$\vec{\zeta} \equiv \vec{x} - \frac{1}{2}\hat{i}\,\eta_1$$

and the integral over $\mathscr{R}^{c}_{ijkl}$ was obtained simply by changing the name of the dummy var-variable of integration from $\vec{y}'$ to $\vec{y}$; hence,

$$\mathscr{R}^{c}_{ijkl}(\vec{y},\vec{\eta},\tau) = 2U(y_2)\left[\delta_{1i}\overline{u_j(\vec{y},t)u_k(\vec{y}+\vec{\eta},t+\tau)u_l(\vec{y}+\vec{\eta},t+\tau)} + \delta_{1j}\overline{u_i(\vec{y},t)u_k(\vec{y}+\vec{\eta},t+\tau)u_l(\vec{y}+\vec{\eta},t+\tau)}\right] \tag{18}$$

Because the distance $|\vec{\eta}|$ over which the correlations $\mathscr{R}^{+}_{ijkl}$ and $\mathscr{R}^{c}_{ijkl}$ are nonzero is certainly much smaller than the region of turbulence, we can always suppose that the observation point $\vec{x}$ is sufficiently far away from the flow so that

$$|\vec{x} - \vec{y}| >> |\vec{\eta}|$$

Therefore, upon expanding the integrands in equation (17) and neglecting terms of order $|\vec{\eta}|/|\vec{x} - \vec{y}|$, we obtain

$$I_\omega(\vec{x}) = \frac{\omega^4\rho_o}{32\pi^3 c_o^5}\,\mathscr{Re}\int \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)(x_l - y_l)}{|\vec{x} - \vec{y}|^6}\iint e^{i\omega\left[\frac{(\vec{x}-\vec{y})\cdot\vec{\eta}}{c_o|\vec{x}-\vec{y}|} - \tau\right]}\left[\mathscr{R}^{+}_{ijkl}(\vec{y},\vec{\eta},\tau) + \mathscr{R}^{c}_{ijkl}(\vec{y},\vec{\eta},\tau)\right]d\vec{\eta}\,d\tau\,d\vec{y} \tag{19}$$

where we have used the fact that

$$|\vec{x} - \vec{y}| - |\vec{x} - \vec{y} - \vec{\eta}| = \frac{(\vec{x} - \vec{y})\cdot\vec{\eta}}{|\vec{x} - \vec{y}|} + \mathcal{O}\left(\frac{|\vec{\eta}|}{|\vec{x} - \vec{y}|}\right)$$

and

$$|\vec{\xi} - \vec{y} + \frac{1}{2}\vec{\eta}| - |\vec{\xi} - \vec{y} - \frac{1}{2}\vec{\eta}| = \frac{(\vec{x} - \vec{y})\cdot\vec{\eta}}{|\vec{x} - \vec{y}|} + \mathcal{O}\left(\frac{|\vec{\eta}|}{|\vec{x} - \vec{y}|}\right)$$

EXPANSION FOR LOW EDDY MACH NUMBER; CONVECTIVE AMPLIFICATION

We realize that, if the mean flow were zero, the correlation functions $\mathscr{R}^{+}_{ijkl}(\vec{y},\vec{\eta},\tau)$ and $\mathscr{R}^{c}_{ijkl}(\vec{y},\vec{\eta},\tau)$ would decay so rapidly with η that we could (as we shall see subsequently) neglect variations in

$$\exp\left\{i\omega\left[\frac{(\vec{x} - \vec{y})\cdot\vec{\eta}}{|\vec{x} - \vec{y}|c_o} - \tau\right]\right\}$$

over the range of $\vec{\eta}$ where $\mathscr{R}^{+}_{ijkl}$ and $\mathscr{R}^{c}_{ijkl}$ are nonzero. However, in the case where there is a mean flow (see ref. 9), the correlation at the points along the direction of flow can extend over long distances since the eddies are moving with the flow and their decay time is quite long. In order to compensate for this we introduce a set of coordinates which "move with the turbulent eddies." Thus, we introduce the new variable

$$\vec{\xi} \equiv \vec{\eta} - \vec{U}_c(y_2)\tau \tag{20}$$

where $\vec{U}_c$, which is in the y_1-direction, is the convection velocity of the eddy (ref. 10) and following Ffowcs Williams we define the moving axis correlation function by

$$\mathscr{R}_{ijkl}(\vec{y}, \vec{\xi}, \tau) \equiv \mathscr{R}^{+}_{ijkl}(\vec{y}, \vec{\eta}, \tau) + \mathscr{R}^{c}_{ijkl}(\vec{y}, \vec{\eta}, \tau) \tag{21}$$

Upon introducing the change of variable (20) into the integral of equation (19) and noting that the Jacobian of the transformation is unity, we get

$$I_\omega(\vec{x}) = \frac{\omega^4 \rho_o}{32\pi^3 c_o^5} \mathcal{R}e \int \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)(x_l - y_l)}{|\vec{x} - \vec{y}|^6} \int e^{-i\omega(1-M_c \cos\theta)\tau} \int \exp\left[\frac{i\omega}{c_o} \frac{(\vec{x} - \vec{y})\cdot\vec{\xi}}{|\vec{x} - \vec{y}|}\right] \mathscr{R}_{ijkl}(\vec{y}, \vec{\xi}, \tau) d\vec{\xi}\, d\tau\, d\vec{y} \tag{22}$$

where $M_c \equiv |\vec{U}_c|/c_o$ is the convection Mach number and

$$\cos\theta \equiv \frac{(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|} \cdot \frac{\vec{U}_c}{|\vec{U}_c|}$$

is the cosine of the angle between the direction of mean flow and the direction of observation.

Now let l denote a typical correlation length of the turbulence in the moving frame and let M_e be Mach number characteristic of the turbulent velocity fluctuations. It is shown in reference 11 (see sec. 3, eq. (17)) that

$$\frac{\omega}{c_o} l \sim M_e$$

Now in naturally occurring turbulent flows (i.e., without artifically induced turbulence) in general and jet flows in particular the turbulent velocity is about one-tenth and certainly not more than two-tenths of the mean flow velocity (ref. 12). Consequently, for subsonic flows

$$\frac{\omega l}{c_o} << 1$$

This shows that the exponential in the innermost integral of equation (22) is approximately equal to unity for $|\vec{\xi}| < l$. But by definition $\mathscr{R}_{ijkl}(\vec{y}, \vec{\xi}, \tau) \approx 0$ for $|\vec{\xi}| > l$. Hence, we can set this exponential equal to unity to obtain the approximate expression

$$I_\omega(\vec{x}) = \frac{\omega^4 \rho_o}{32\pi^3 c_o^5} \mathcal{R}e \int \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)(x_l - y_l)}{|\vec{x} - \vec{y}|^6} \int e^{-i\omega(1-M_c \cos\theta)\tau} \int \mathscr{R}_{ijkl}(\vec{y}, \vec{\xi}, \tau) d\vec{\xi}\, d\tau\, d\vec{y} \tag{23}$$

Following Ribner (ref. 4) we let $I_\omega(\vec{x}|\vec{y})$ denote the intensity at the point x due to the sound emitted from a unit volume at the point y. Then

$$I_\omega(\vec{x}) = \int I_\omega(\vec{x}|\vec{y}) d\vec{y}$$

and it follows from equation (23) that

$$I_\omega(\vec{x}|\vec{y}) = \frac{\omega^4 \rho_o}{32\pi^3 c_o^5} \frac{r_i r_j r_k r_l}{r^6} \mathcal{R}e \int e^{-i\omega(1-M_c \cos\theta)\tau} \int \mathscr{R}_{ijkl}(\vec{y}, \vec{\xi}, \tau) d\vec{\xi}\, d\tau \tag{24}$$

where

$$\vec{r} \equiv \vec{x} - \vec{y}$$

is the vector joining the source point and the observation point and

$$r \equiv |\vec{r}|$$

is the magnitude of the vector.

It follows from equations (16), (18), and (21) that

$$\int \mathcal{R}_{ijkl}(\vec{y}, \vec{\xi}, \tau)d\vec{\xi} = \int \widetilde{R}_{ijkl}(\vec{y}, \vec{\xi}, \tau)d\xi + \int U\left(\vec{y} - \frac{\vec{\xi}}{2}\right)U\left(\vec{y} + \frac{\vec{\xi}}{2}\right)\left[\delta_{1i}\delta_{1k}\widetilde{R}_{jl}(\vec{y}, \vec{\xi}, \tau) + \delta_{1j}\delta_{1l}\widetilde{R}_{ik}(\vec{y}, \vec{\xi}, \tau) + 2\delta_{1j}\delta_{1k}R_{il}(\vec{y}, \vec{\xi}, \tau)\right]d\vec{\xi}$$

$$+ 2U(\vec{y}) \int \left[\delta_{1i}\widetilde{R}_{j,kl}(\vec{y}, \vec{\xi}, \tau) + \delta_{1j}\widetilde{R}_{i,kl}(\vec{y}, \vec{\xi}, \tau)\right]d\vec{\xi} \tag{25}$$

where

$$\widetilde{R}_{ijkl}(\vec{y}, \vec{\xi}, \tau) \equiv \overline{u_i'u_j'u_k''u_l''} - \overline{u_i'u_j'}\ \overline{u_k'u_l'} \qquad \vec{y} = \frac{y' + y''}{2} + \hat{i}\left(\frac{y_1' - y_1''}{2}\right) \tag{26}$$

$$\widetilde{R}_{ij}(\vec{y}, \vec{\xi}, \tau) \equiv \overline{u_i'u_j'} \qquad \vec{y} = \frac{y' + y''}{2} + \hat{i}\left(\frac{y_1' - y_1''}{2}\right) \tag{27}$$

$$\widetilde{R}_{i,jk}(\vec{y}, \vec{\xi}, \tau) \equiv \overline{u_i(\vec{y}, t)u_j(\vec{y} + \vec{\eta}, t + \tau)u_k(\vec{y} + \vec{\eta}, t + \tau)} \tag{28}$$

the first two integrals on the right-hand side of equation (25) represent the self-noise and the shear noise, respectively (refs. 3 and 13). This terminology was introduced by Lilley in reference 3 to indicate that the former term represents noise generated by turbulent-turbulent interactions whereas the latter term represents noise generated by turbulent-shear interactions. The last integral represents a coupling between the shear noise and the self-noise. We shall now show, under the relatively mild restriction that the turbulence is locally homogeneous and incompressible, that the coupling term in equation (25) vanishes.

LOCALLY HOMOGENEOUS TURBULENCE; VANISHING OF COUPLING TERM

The theory of incompressible locally homogeneous turbulence is developed in reference 8 and in chapter 4 of reference 14. The assumption of local homogeneity implies that any two-time correlation function involving the two points $\vec{y}$ and $\vec{z}$, say $Q(\vec{y}, \vec{z}, \tau)$, is a function only of $\vec{\eta} = \vec{z} - \vec{y}$ and τ. Thus

$$Q(\vec{y}, \vec{z}, \tau) = q(\vec{z} - \vec{y}, \tau)$$

and this implies that

$$\left(\frac{\partial Q}{\partial y_i}\right)_{\vec{z}} = -\left(\frac{\partial Q}{\partial z_i}\right)_{\vec{y}} \tag{29}$$

On the other hand,

$$\left(\frac{\partial Q}{\partial z_i}\right)_{\vec{y}} = \left(\frac{\partial Q}{\partial y_j}\right)_{\vec{\xi}}\left(\frac{\partial y_j}{\partial z_i}\right)_{\vec{y}} + \left(\frac{\partial Q}{\partial \xi_j}\right)_{\vec{y}}\left(\frac{\partial \xi_j}{\partial z_i}\right)_{\vec{y}}$$

Hence it follows from equation (20) that

$$\left(\frac{\partial Q}{\partial z_i}\right)_{\vec{y}} = \left(\frac{\partial Q}{\partial \xi_i}\right)_{\vec{y}}$$

and by using equation (29), we find that

$$\left(\frac{\partial Q}{\partial y_i}\right)_{\vec{z}} = -\left(\frac{\partial Q}{\partial \xi_i}\right)_{\vec{y}} \tag{30}$$

We shall now assume that the flow is incompressible and locally homogeneous. Then, since $\partial U/\partial y_1 = 0$, the continuity equation is $\partial u_j/\partial y_j = 0$. Hence, if f' is any function of the turbulent velocities at $\vec{z}$ and $t + \tau$,

$$\left[\frac{\partial}{\partial y_j}(u_j f')\right]_{\vec{z}} = 0$$

which becomes upon taking the time average

$$\left(\frac{\partial}{\partial y_j}\overline{u_j f'}\right)_{\vec{z}} = 0$$

But since the turbulence is locally homogeneous, it follows from equation (30) that

$$\left(\frac{\partial}{\partial \xi_j}\overline{u_j f'}\right)_{\vec{y}} = 0$$

and, therefore, that for any volume V

$$\int_V \xi_i \frac{\partial}{\partial \xi_j} \overline{u_j f'}\, d\vec{\xi} = 0$$

Hence

$$\int_V \frac{\partial}{\partial \xi_j} (\xi_i \overline{u_j f'}) d\vec{\xi} - \int_V \overline{u_i f'}\, d\vec{\xi} = 0$$

Upon using the divergence theorem, this becomes

$$\int_V \overline{u_i f'}\, d\vec{\xi} = \int_S n_j \xi_i \overline{u_j f'}\, dS$$

where S is the surface of V and n_i is the i^{th} component of the outward unit normal. It follows that, if $\overline{u_i f'}$ goes to zero faster than $|\xi|^{-3}$ as $|\xi| \to \infty$, the surface integral will vanish in this limit and we obtain

$$\int \overline{u_i f'}\, d\vec{\xi} = 0 \tag{31}$$

the integral being carried out over all space. It is shown in reference 15 that

$$\overline{u_i(\vec{y},t)u_j(\vec{y} + \vec{\eta},t)u_k(\vec{y} + \vec{\eta},t)} = \mathcal{O}\left(|\vec{\eta}|^{-4}\right) \text{ as } |\vec{\eta}| \to \infty$$

Since, by equation (16), $\vec{\eta} = \vec{\xi}$ at $\tau = 0$, we can write from equation (28)

$$\widetilde{R}_{i,jk}(\vec{y}, \vec{\xi}, \tau) = \mathcal{O}\left(|\vec{\xi}|^{-4}\right) \text{ as } |\vec{\xi}| \to \infty \text{ as } \tau = 0 \tag{32}$$

And since the two observation points always correspond to the same points on the moving eddy for all τ when $\vec{\xi}$ is fixed, it follows that the velocities at these points will always be at least as well correlated at $\tau = 0$ as for any other time delay. Hence we can write

$$|\widetilde{R}_{i,jk}(\vec{y}, \vec{\xi}, \tau)| \leq |\widetilde{R}_{i,jk}(\vec{y}, \vec{\xi}, 0)|$$

and it follows from equation (32) that

$$\tilde{R}_{i,jk}(\vec{y}, \vec{\xi}, \tau) = \mathcal{O}|\vec{\xi}|^{-4} \quad \text{as} \quad |\vec{\xi}| \to \infty$$

Now we can take $f' = u_j(\vec{y} + \vec{\eta}, t + \tau)u_k(\vec{y} + \vec{\eta}, t + \tau)$ in equation (31) to show that

$$\int \tilde{R}_{i,jk}\, d\vec{\xi} = 0$$

But this shows that the last integral in equation (25) is zero. After changing the names of the dummy indices, we can write equation (24) as

$$I_\omega(\vec{x}|\vec{y}) = \frac{\omega^4 \rho_o}{32\pi^3 c_o^5} \frac{r_i r_j r_k r_l}{r^6} \mathcal{Re} \int e^{-i\omega(1-M_c \cos\theta)\tau} \int \tilde{R}_{ijkl}(\vec{y}, \vec{\xi}, \tau) d\vec{\xi}\, d\tau$$

$$+ \frac{\omega^4 \rho_o}{8\pi^3 c_o^5} \frac{r_1 r_j r_l}{r^6} \mathcal{Re} \int e^{-i\omega(1-M_c\cos\theta)\tau} \int U\left(y_2 - \frac{\xi_2}{2}\right) U\left(y_2 + \frac{\xi_2}{2}\right) \tilde{R}_{jl}(\vec{y}, \vec{\xi}, \tau) d\vec{\xi}\, d\tau \tag{33}$$

The first term in this equation is the self-noise term and the second is the shear noise term. Ribner (refs. 4 and 13) showed that this decomposition occurs for $\int I_\omega(\vec{x}|\vec{y}) d\omega$, the overall intensity at $\vec{x}$, (without including the effect of the moving sources) provided the turbulence correlations factor into a time-dependent part and a space-dependent part and that the turbulence is locally homogeneous and isotropic. We have shown here that this decomposition actually occurs under the less restrictive assumption that the turbulence is locally homogeneous.

Equation (33) cannot be further simplified without introducing additional assumptions about the turbulence.

REPRESENTATION OF FOURTH-ORDER CORRELATIONS IN TERMS OF SECOND-ORDER CORRELATIONS

It is argued by Batchelor (ref. 8) that the part of the joint probability distribution of the velocities at a fixed time associated with the energy bearing eddies is approximately normal at least in so far as the velocities at two points are concerned. This approximation is better for some purposes than others. Thus, (ref. 8, p. 176) it gives reasonably accurate predictions about the relation between the second- and fourth-order correlations. This relation is (ref. 8, eq. (8.3.11) found to be (see ref. 16, sec. 2.1.7 for derivation)

$$\overline{u_i'u_j'u_k''u_l''} = \overline{u_i'u_j'}\;\overline{u_k''u_l''} + \overline{u_i'u_k''}\;\overline{u_j'u_l''} + \overline{u_i'u_l''}\;\overline{u_j'u_k''} \qquad \text{at} \quad \tau = 0 \tag{34}$$

But by extending the reasoning used by Batchelor in section 8.2, we can argue that, when the velocity correlations are separated in time as well as in space, their correlation in the moving frame will be subject to even more random influences from the neighboring flow than when they occur at the same time. In accordance with the central limit theorem, these influences will tend to make the joint probability distribution more normal. Hence we expect equation (34), written in the moving frame, to be even more valid when $\tau \neq 0$ and, in view of equation (30) and (31), we can now write

$$\widetilde{R}_{ijkl}(\vec{y}, \vec{\xi}, \tau) = \widetilde{R}_{ik}\widetilde{R}_{jl} + \widetilde{R}_{il}\widetilde{R}_{jk} \tag{35}$$

Upon inserting this into equation (32) and changing the dummy indices, we obtain

$$I_\omega(\vec{x}|\vec{y}) = \frac{\omega^4 \rho_o}{16\pi^3 c_o^5} \frac{r_i r_j r_k r_l}{r^6} \, Re \int e^{-i\omega(1-M_c \cos\theta)\tau} \int \widetilde{R}_{ik}\widetilde{R}_{jl} \, d\vec{\xi} \, d\tau$$

$$+ \frac{\omega^4 \rho_o}{8\pi^3 c_o^5} \frac{r_1^2 r_j r_l}{r^6} \, Re \int e^{-i\omega(1-M_c \cos\theta)\tau} \int U\left(y_2 - \frac{\xi_2}{2}\right) U\left(y_2 + \frac{\xi_2}{2}\right) \widetilde{R}_{jl} \, d\vec{\xi} \, d\tau$$

Having derived this equation in the $\vec{\xi}$ coordinate system, we can now return to the $\vec{\eta}$ coordinates and carry out the integration with respect to this variable. We introduce the fixed-axis correlation R_{ij} by

$$R_{ij}(\vec{y}, \vec{\eta}, \tau) \equiv \widetilde{R}_{ij}(\vec{y}, \vec{\xi}, \tau) = \overline{u_i'u_j''} \qquad \vec{y} \equiv \frac{\vec{y}' + \vec{y}''}{2} + \hat{i}\,\frac{\vec{y}_1' - \vec{y}_1''}{2}$$

and the equation can be written in terms of the fixed-axis correlations as

$$I_\omega(\vec{x}|\vec{y}) = \frac{\omega^4 \rho_o}{32\pi^3 c_o^5} \frac{r_i r_j r_k r_l}{r^6} \mathcal{R}e \int e^{-i\omega(1-M_c \cos\theta)\tau} \int R_{ik}(\vec{y},\vec{\eta},\tau) R_{jl}(\vec{y},\vec{\eta},\tau) d\vec{\eta}\, d\tau$$

$$+ \frac{\omega^4 \rho_o}{8\pi^3 c_o^5} \frac{r_1 r_j r_l}{r^6} \mathcal{R}e \int e^{-i\omega(1-M_c \cos\theta)\tau} \int U\left(y_2 - \frac{\eta_2}{2}\right) U\left(y_2 - \frac{\eta_2}{2}\right) R_{jl}(\vec{y},\vec{\eta},\tau) d\vec{\eta}\, d\tau \tag{36}$$

In order to simplify equation (36), we now consider a specialization of the turbulence field.

AXISYMMETRIC TURBULENCE

Ribner assumed that the turbulence is locally isotropic. However, the turbulence in a jet is not necessarily isotropic. For example, the measurements of Jones (ref. 5, see fig. 5) show that the correlation $R_{1111}(\vec{y},0,0)$ of the velocities in the direction of flow differs noticeably from the nearly equal correlations $R_{2222}(\vec{y},0,0)$ and $R_{3333}(\vec{y},0,0)$ in the directions transverse to the flow (also see ref. 11, sec. 3). These differences could have a significant effect on the sound emitted from the jet. A better assumption would be that the turbulence is locally axisymmetric. Axisymmetric turbulence was introduced by Batchelor in reference 6. He defined axisymmetric turbulence as being any turbulence which satisfies the following criteria. "The mean value of any function of the velocities and their derivatives must . . . . be independent[2] of arbitrary rotations of the axes about a given direction (which in all practical realizations will be the direction of mean flow) and of reflections in planes through that direction." He later includes the condition that it be independent of reflections in planes perpendicular to that direction.

It follows from Batchelor's definition that any two-point correlation of the velocities and their derivatives (with zero time delay) is an axisymmetric tensor (ref. 7). Thus, in particular

$$\overline{u_i(\vec{y}',t) \frac{\partial^n u_j}{\partial t^n}(\vec{y}'',t)} \qquad \text{for } n = 0, 1, 2, \ldots$$

is an axisymmetric tensor.

[2]The function should have the same form.

Now for any stationary process

$$\frac{\partial^n}{\partial \tau^n} \overline{u_i(\vec{y}',t)u_j(\vec{y}'',t+\tau)}\bigg]_{\tau=0} = \overline{u_i(\vec{y}',t)\frac{\partial^n u_j}{\partial t^n}(\vec{y}'',t)}$$

But by expanding in a Taylor series

$$\overline{u_i(\vec{y}',t)u_j(\vec{y}'',t+\tau)} = \sum_{n=0}^{\infty} \frac{\tau^n}{n!}\left[\frac{\partial^n}{\partial \tau^n} \overline{u_i(\vec{y}',t)u_j(\vec{y}'',t+\tau)}\right]_{\tau=0}$$

so that

$$R_{ij}(\vec{y},\vec{\eta},\tau) \equiv \overline{u_i(\vec{y}',t)u_j(\vec{y}'',t+\tau)} \qquad \vec{y} \equiv \frac{\vec{y}'+\vec{y}''}{2} + \hat{i}\left(\frac{\vec{y}_1'-\vec{y}_1''}{2}\right) \tag{37}$$

is a linear combination of axisymmetric tensors and is, therefore, itself an axisymmetric tensor. Thus, when the axis of symmetry is taken in the flow direction (i.e., the η_1-direction), the correlation R_{ij} is of the form (ref. 6)

$$R_{ij}(\vec{y},\vec{\eta},\tau) = A\eta_i\eta_j + B\delta_{ij} + C\delta_{1i}\delta_{1j} + D\delta_{1i}\eta_j + E\delta_{1j}\eta_i$$

where A, B, C, D, and E are arbitrary functions of $\vec{y}$, τ, $|\vec{\eta}| = \sqrt{\eta_1^2+\eta_2^2+\eta_3^2}$, and η_1.[3] Notice that substituting equation (20) into this equation and recalling that $\vec{U}_c$ is in the y_1-direction show that the moving axis correlation tensor (27) is also an axisymmetric tensor.

Since the flow is incompressible,

$$\frac{\partial u_i(y',t)}{\partial y_i'} = 0$$

Hence,

---

[3]In ref. 6, Batchelor uses r in place of our η and μr in place of our η_1.

$$\frac{\partial}{\partial y_i'} \overline{u_i(\vec{y}',t)u_j(\vec{y}'',t+\tau)} = 0$$

But since the flow is locally homogeneous, this shows that (see eq. (37))

$$\frac{\partial}{\partial \eta_i} R_{ij}(\vec{y},\vec{\eta},\tau) = 0 \tag{38}$$

and R_{ij} is solenoidal (i.e., it has zero divergence).

Since the flow is locally homogeneous, the second-order correlations must be invariant with respect to translation of the $\vec{\eta}$ coordinate. Consequently, these correlations cannot change if we replace $\vec{y}$ by $\vec{y} - \hat{i}\,\eta$; since $\vec{y}' = \vec{y} + \frac{1}{2}(\hat{i},\eta_1 - \vec{\eta})$ and $\vec{y}'' = \vec{y} + \frac{1}{2}(\hat{i},\eta_1 + \vec{\eta})$, it follows that

$$\overline{u_i(\vec{y}',t)u_j(\vec{y}'',t+\tau)} = \overline{u_i\left(\vec{y} - \frac{1}{2}\hat{i}\,\vec{\eta}_1 - \frac{1}{2}\vec{\eta},t\right)u_j\left(\vec{y} - \frac{1}{2}\hat{i}\,\eta_1 + \frac{1}{2}\vec{\eta},t+\tau\right)}$$

Also since the turbulence is stationary in time, the correlations must be independent of translations of the time coordinate. Hence,

$$\overline{u_i\left(\vec{y} - \frac{1}{2}\hat{i}\,\eta_1 - \frac{1}{2}\vec{\eta},t\right)u_j\left(\vec{y} - \frac{1}{2}\hat{i}\,\eta_1 + \frac{1}{2}\vec{\eta},t+\tau\right)} = \overline{u_i\left(\vec{y} - \frac{1}{2}\hat{i}\,\eta_1 - \frac{1}{2}\vec{\eta},t-\tau\right)u_j\left(\vec{y} - \frac{1}{2}\hat{i}\,\eta_1 + \frac{1}{2}\vec{\eta},t\right)}$$

and in view of equation (36), this shows that

$$R_{ij}(\vec{y},\vec{\eta},\tau) = R_{ji}(\vec{y},-\vec{\eta},-\tau) \tag{39}$$

We now introduce the axisymmetric tensors $R_{ij}^{\pm}(\vec{y},\vec{\eta},\tau)$ by

$$R_{ij}^{\pm}(\vec{y},\vec{\eta},\tau) = \frac{1}{2}\left[R_{ij}(\vec{y},\vec{\eta},\tau) \pm R_{ij}(\vec{y},\vec{\eta},-\tau)\right] \tag{40}$$

Then

$$R_{ij}^{\pm} = A^{\pm}\eta_i\eta_j + B^{\pm}\delta_{ij} + C^{\pm}\delta_{1i}\delta_{1j} + D^{\pm}\delta_{1i}\eta_j + D^{\pm}\delta_{1j}\eta_i$$

where

$$2G^{\pm}(\tau) = G(\tau) \pm G(-\tau)$$

for

$$G = A, B, C, D, \text{ or } E \tag{41}$$

and the dependence on the other variables besides τ has been suppressed for simplicity. Then equations (39) and (40) show that

$$R_{ij}^{+}(\vec{y}, \vec{\eta}, \tau) = R_{ji}^{+}(\vec{y}, -\vec{\eta}, \tau) \tag{42}$$

$$R_{ij}^{-}(\vec{y}, \vec{\eta}, \tau) = -R_{ji}^{-}(\vec{y}, -\vec{\eta}, \tau) \tag{43}$$

and equations (38) and (40) show that R_{ij}^{+} and R_{ij}^{-} are solenoidal.

It follows by the same argument as used in reference 6 (also see ref. 7, p. 570) that, since R_{ij}^{+} is an axisymmetric solenoidal tensor which satisfies the symmetry condition (42), its coefficients A^{+}, B^{+}, and C^{+} are functions of $|\vec{\eta}|$ and are even functions of η_1; and its coefficients D^{+} and E^{+} are equal to one another and are odd functions of η_1.

A similar argument can be used to deduce the dependence of the coefficients of R_{ij}^{-} on η_1 from the skew-symmetry conditions (43). Thus, by taking $i \neq j$ and $i, j \neq 1$ we see from equation (43) that $A^{-}(-\eta_1) = -A^{-}(\eta_1)$. By taking $i = j$ and $i, j \neq 1$, it follows that $B^{-}(-\eta_1) = -B^{-}(\eta_1)$. By taking $i = 1$ and $j \neq 1$, we find that

$$D^{-}(-\eta_1) = E^{-}(\eta_1) \tag{44}$$

Finally by taking $i = j = 1$, we find that $C^{-}(-\eta_1) = -C^{-}(\eta_1)$.

Now it can be shown by the same argument as in reference 12 that the requirement that R_{ij}^{-} be solenoidal implies that

$$\frac{1}{\eta}\frac{\partial B^{-}}{\partial \mu} + \mu \frac{\partial C^{-}}{\partial \eta} + \frac{1 - \mu^2}{\eta}\frac{\partial C^{-}}{\partial \mu} + 3D^{-} + \eta \frac{\partial D^{-}}{\partial \eta} + E^{-} = 0$$

where $\eta = |\vec{\eta}|$ and we have put $\eta_1 = \eta\mu$. Since B^{-} and C^{-} are odd functions of η_1 (and, therefore, of μ), it is easy to see from this that

$$3D^{-} + \eta \frac{\partial D^{-}}{\partial \eta} + E^{-}$$

is an even function of η_1. Hence, if we put $D_o^- = D^-(\eta_1) - D^-(-\eta_1)$ and $E_o^- = E^-(\eta_1) - E^-(-\eta_1)$, it follows that

$$3D_o^- + \eta \frac{\partial D_o^-}{\partial \eta} + E_o^- = 0$$

But it follows from equation (44) that $E_o^- = -D_o^-$. Hence,

$$2D_o^- + \eta \frac{\partial D_o^-}{\partial \eta} = 0$$

which shows that

$$D_o^- \eta^2 = f(\mu)$$

But as indicated in reference 6, regularity requires that we take $f(\mu) = 0$. Therefore, $D^-(\eta_1) = D^-(-\eta_1)$ and equation (44) implies that $D^-(\eta_1) = E^-(\eta_1)$. By adding and subtracting equation (40) and collecting results, we can conclude that

$$R_{ij}(\vec{y}, \vec{\eta}, \tau) = R_{ij}^+(\vec{y}, \vec{\eta}, \tau) + R_{ij}^-(\vec{y}, \vec{\eta}, \tau) \tag{45}$$

where

$$R_{ij}^+ = A^+\eta_i\eta_j + B^+\delta_{ij} + C^+\delta_{1i}\delta_{1j} + D^+(\delta_{1i}\eta_j + \delta_{1j}\eta_i) \tag{46}$$

is an even function of τ (which follows from eqs. (40) and (41)) whose coefficients A^+, B^+, and C^+ are functions of $\vec{y}$, τ, and $\eta = \sqrt{\eta_1^2 + \eta_2^2 + \eta_3^2}$ and even functions of η_1 and D^+ is a function of $\vec{y}$, τ, and η and an odd function of η_1. Also

$$R_{ij}^- = A^-\eta_i\eta_j + B^-\delta_{ij} + C^-\delta_{1i}\delta_{1j} + D^-(\delta_{1i}\eta_j + \delta_{1j}\eta_i) \tag{47}$$

is an odd function of τ whose coefficients A^-, B^-, and C^- are functions of $\vec{y}$, τ, and η and odd functions of η_1; and D^- is a function of $\vec{y}$, τ, and η and an even function of η_1. Notice that in obtaining this representation of the correlation tensor it was only necessary to use equations (38) and (39). But it follows from these latter two equations and equation (20) that the moving frame correlation tensor $\tilde{R}_{ij}$ must satisfy the same two equations with $\vec{\eta}$ replaced by $\vec{\xi}$. Hence, this implies that $\tilde{R}_{ij}$ can be expressed in the same form as equations (45) to (47) with $\vec{\xi}$ replaced by $\vec{\eta}$.

It can be seen from equations (46) and (47) that $R_{ij}^{\pm} = R_{ji}^{\pm}$.

AXISYMMETRIC-QUADRUPOLE CORRELATIONS AND SOUND DIRECTIVITY

We shall now use the symmetry condition derived in the preceding section to simplify equation (35). It follows from equation (45) that

$$\mathcal{R}e \int_{-\infty}^{\infty} e^{-i\omega(1-M_c \cos\theta)\tau} R_{ik}R_{jl}\, d\tau = \mathcal{R}e \int_{-\infty}^{\infty} e^{-i\omega(1-M_c \cos\theta)\tau}\left[R_{ik}^{+}R_{jl}^{+} + R_{ik}^{-}R_{jl}^{-} + R_{ik}^{+}R_{jl}^{-} + R_{ik}^{-}R_{jl}^{+}\right] d\tau$$

Since $R_{ik}^{+}R_{jl}^{-} + R_{ik}^{-}R_{jl}^{+}$ is an odd function of τ, we get

$$\int_{-\infty}^{\infty} e^{-i\omega(1-M_c \cos\theta)\tau} (R_{ik}^{+}R_{jl}^{-} + R_{ik}^{-}R_{jl}^{+}) d\tau = -i \int_{-\infty}^{\infty} \sin\left[\omega(1 - M_c \cos\theta)\tau\right] R_{ik}^{+}R_{jl}^{-} + R_{ik}^{-}R_{jl}^{+}\, d\tau.$$

Hence, the left-hand side is purely imaginary and we find

$$\mathcal{R}e \int_{-\infty}^{\infty} e^{-i\omega(1-M_c \cos\theta)\tau} R_{ik}R_{jl}\, d\tau = \int_{-\infty}^{\infty} e^{-i\omega(1-M_c \cos\theta)\tau}\left(R_{ik}^{+}R_{jl}^{+} + R_{ik}^{-}R_{jl}^{-}\right) d\tau$$

since the integral on the right-hand side is real. In a similar way it follows that

$$\mathcal{R}e \int_{-\infty}^{\infty} e^{-i\omega(1-M_c \cos\theta)\tau} U\left(y_2 - \frac{\eta_2}{2}\right) U\left(y_2 + \frac{\eta_2}{2}\right) R_{jl}\, d\tau = \int_{-\infty}^{\infty} e^{-i\omega(1-M_c \cos\theta)\tau} U\left(y_2 - \frac{\eta_2}{2}\right) U\left(y_2 + \frac{\eta_2}{2}\right) R_{jl}^{+}\, d\tau$$

Equation (36) thereby becomes

$$I_\omega(\vec{x}|\vec{y}) = \frac{\omega^4\rho_o}{16\pi^3 c_o^5}\frac{r_i r_j r_k r_l}{r^6}\int e^{-i\omega(1-M_c\cos\theta)\tau}\int\left(R_{ik}^+R_{jl}^+ + R_{ik}^-R_{jl}^-\right)d\vec{\eta}\, d\tau$$

$$+ \frac{\omega^4\rho_o}{8\pi^3 c_o^5}\frac{r_1 r_j r_l}{r^6}\int e^{-i\omega(1-M_c\cos\theta)\tau}\int U\left(y_2 - \frac{\eta_2}{2}\right)U\left(y_2 + \frac{\eta_2}{2}\right)R_{jl}^+\, d\vec{\eta}\, d\tau \quad (48)$$

We first consider the first term on the right-hand side of equation (48). It follows from equations (46) and (47) that

$$R_{ik}^\pm R_{jl}^\pm\, d\vec{\eta} = \int\left[A^\pm\eta_i\eta_k + B^\pm\delta_{ik} + C^\pm\delta_{1i}\delta_{1k} + D^\pm(\delta_{1i}\eta_k + \delta_{1k}\eta_i)\right]$$

$$\times\left[A^\pm\eta_j\eta_l + B^\pm\delta_{jl} + C^\pm\delta_{1j}\delta_{1l} + D^\pm(\delta_{1j}\eta_l + \delta_{1l}\eta_j)\right]d\vec{\eta} \quad (49)$$

Notice that all the products of the coefficients $A^\pm$, $B^\pm$, and so forth which occur in these integrands are even functions of η_2 and η_3 and either even or odd functions of η_1. (In fact they are all even except those terms which consist of products of $D^\pm$ with either $A^\pm$, $B^\pm$, or $C^\pm$.) But since any integral of an odd function of η_i for $i = 1,2,3$ is equal to zero, it can be seen by inspection that only terms which have two pairs of equal indices can contribute to the integral. Thus, the integrals must be of the form

$$\int R_{ik}^\pm R_{jl}^\pm\, d\vec{\eta} = a_{ik}^\pm\delta_{ij}\delta_{kl} + b_{ij}^\pm\delta_{ik}\delta_{jl} + c_{ij}^\pm\delta_{il}\delta_{jk} \qquad \text{(no sum on } i,\ j,\ k,\text{ or } l)$$

Upon interchanging names of dummy indices, we find that

$$r_i r_j r_k r_l\int R_{ik}^\pm R_{jl}^\pm\, d\vec{\eta} = a_{ik}^\pm r_i^2 r_k^2 + b_{ij}^\pm r_i^2 r_j^2 + c_{ij}^\pm r_i^2 r_j^2 = \left(a_{ij}^\pm + b_{ij}^\pm + c_{ij}^\pm\right)r_i^2 r_j^2 \quad (50)$$

But this shows that the coefficient of $r_i^2 r_j^2$ in the summation is just the sum of all terms in $\int R_{ik}^\pm R_{jl}^\pm\, d\vec{\eta}$ which have two indices equal to i and the other two equal to j. Thus, employing the symmetry condition that $R_{ij}^\pm = R_{ji}^\pm$ as noted immediately after equation (47), we find by inspection of equation (49) that the coefficient of $r_i^2 r_j^2$ in equation (50) is

$$4\int\left(R_{ij}^\pm\right)^2 d\vec{\eta} + 2\int R_{ii}^\pm R_{jj}^\pm\, d\vec{\eta} \qquad \text{if } i \neq j \text{ (no sum on } i,j)$$

or

$$\int \left(R_{ii}^{\pm}\right)^2 d\vec{\eta} \qquad \text{if } i = j \text{ (no sum on } i)$$

Equation (50) now becomes

$$\begin{aligned} r_i r_j r_k r_l \int R_{ik}^{\pm} R_{jl}^{\pm}\, d\vec{\eta} = r_1^4 \int \left(R_{11}^{\pm}\right)^2 d\vec{\eta} + r_2^4 \int \left(R_{22}^{\pm}\right)^2 d\vec{\eta} + r_3^4 \int \left(R_{33}^{\pm}\right)^2 d\vec{\eta} \\ + 2r_1^2 r_3^2 \int \left(2R_{13}^{\pm} R_{13}^{\pm} + R_{11}^{\pm} R_{33}^{\pm}\right) d\vec{\eta} \\ + 2r_1^2 r_2^2 \int \left(2R_{12}^{\pm} R_{12}^{\pm} + R_{11}^{\pm} R_{22}^{\pm}\right) d\vec{\eta} \\ + 2r_2^2 r_3^2 \int \left(2R_{23}^{\pm} R_{23}^{\pm} + R_{22}^{\pm} R_{33}^{\pm}\right) d\vec{\eta} \end{aligned} \tag{51}$$

Upon carrying out the integration with respect to $\vec{\eta}$ in terms of polar coordinates, we find that

$$\int \left(R_{22}^{\pm}\right)^2 d\vec{\eta} = \int \left(R_{33}^{\pm}\right)^2 d\vec{\eta}$$

$$\int \left(2R_{13}^{\pm} R_{13}^{\pm} + R_{11}^{\pm} R_{33}^{\pm}\right) d\vec{\eta} = \int \left(2R_{12}^{\pm} R_{12}^{\pm} + R_{11}^{\pm} R_{22}^{\pm}\right) d\vec{\eta}$$

$$\int \left(2R_{23}^{\pm} R_{23}^{\pm} + R_{22}^{\pm} R_{33}^{\pm}\right) d\vec{\eta} = \int \left(R_{22}^{\pm}\right)^2 d\vec{\eta}$$

Hence equation (51) becomes

$$\begin{aligned} r_i r_j r_k r_l \int R_{ik}^{\pm} R_{jl}^{\pm}\, d\vec{\eta} = r_1^4 \int \left(R_{11}^{\pm}\right)^2 d\vec{\eta} + 2r_1^2\left(r_2^2 + r_3^2\right) \int \left(2R_{12}^{\pm} R_{12}^{\pm} + R_{11}^{\pm} R_{22}^{\pm}\right) d\vec{\eta} \\ + \left(r_2^2 + r_3^2\right)^2 \int \left(R_{22}^{\pm}\right)^2 d\vec{\eta} \end{aligned}$$

and since $r_1^2 + r_2^2 + r_3^2 = r^2$, this becomes

$$r_i r_j r_k r_l \int R_{ik}^{\pm} R_{jl}^{\pm} \, d\vec{\eta} = r_1^4 \int \left(R_{11}\right)^2 d\vec{\eta} + 2r_1^2\left(r^2 - r_1^2\right) \int (2R_{12}^{\pm} R_{12}^{\pm} + R_{11}^{\pm} R_{22}^{\pm}) d\vec{\eta}$$

$$+ \left(r^2 - r_1^2\right)^2 \quad \left(R_{22}^{\pm}\right)^2 d\vec{\eta} \qquad (52)$$

This can be expressed in terms of the angle θ between the direction of flow and the direction r/r from the observation point to the source point (see fig. 1). By using the relation

$$r_1 = r \cos \theta \qquad (53)$$

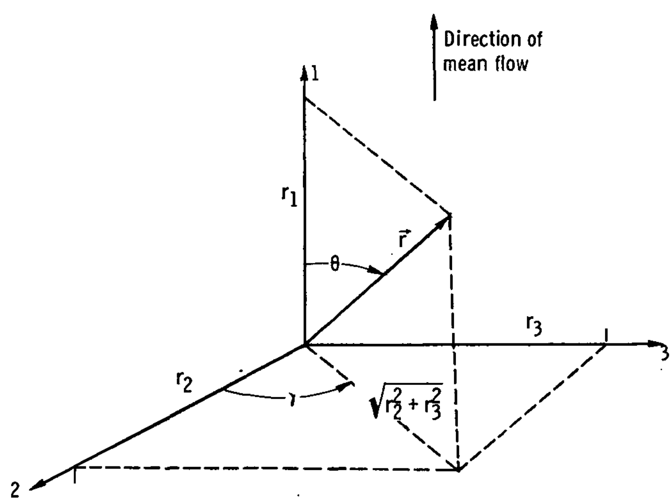

Figure 1. - Coordinates for direction between observation and source points.

equation (52) becomes

$$\frac{r_i r_j r_k r_l}{r^4} \int R_{ik}^{\pm} R_{jl}^{\pm} \, d\vec{\eta} = \cos^4\theta \int \left(R_{11}^{\pm}\right)^2 d\vec{\eta} + 2 \sin^2\theta \cos^2\theta \int (2R_{12}^{\pm} R_{12}^{\pm} + R_{11}^{\pm} R_{22}^{\pm}) d\vec{\eta}$$

$$+ \sin^4\theta \int \left(R_{22}^{\pm}\right)^2 d\vec{\eta} \qquad (54)$$

This completes the treatment of the first term on the right-hand side of equation (48).

Considering now the second term in equation (48), we find from equation (46) that

$$r_j r_l \int U'U''R^+_{jl}\, d\vec{\eta} = r_j r_l \int U'U''A^+\eta_j\eta_l\, d\vec{\eta} + r^2 \int U'U''B^+\, d\vec{\eta} + r_1^2 \int U'U''C^+\, d\vec{\eta} + 2r_1 r_j \int U'U''D^+\eta_j\, d\vec{\eta} \tag{55}$$

where $U' \equiv U\left(y_2 - \frac{\eta_2}{2}\right)$ and $U'' \equiv \left(y_2 + \frac{\eta_2}{2}\right)$. The first integral on the right-hand side of this equation vanishes whenever odd powers of η_1 and η_3 occur. Thus, only terms with $j = l$ will contribute to this integral. The last integral on the right-hand side of equation (55) vanishes unless $j = 1$ since D^+ is an odd function of η_1. Hence equation (55) becomes

$$r_j r_l \int U'U''R^+_{jl}\, d\vec{\eta} = r_1^2 \int U'U''A^+\eta_1^2\, d\vec{\eta} + r_2^2 \int U'U''A^+\eta_2^2\, d\vec{\eta} + r_3^2 \int U'U''A^+\eta_3^2\, d\vec{\eta}$$

$$+ \left(r_1^2 + r_2^2 + r_3^2\right) \int U'U''B^+\, d\vec{\eta} + r_1^2 \int U'U''C^+\, d\vec{\eta} + 2r_1^2 \int U'U''D^+\eta_1\, d\vec{\eta}$$

which can be written as

$$r_j r_l \int U'U''R^+_{jl}\, d\vec{\eta} = r_1^2 \int U'U''R^+_{11}\, d\vec{\eta} + r_2^2 \int U'U''R^+_{22}\, d\vec{\eta} + r_3^2 \int U'U''R^+_{33}\, d\vec{\eta} \tag{56}$$

But it can be seen from figure 1 that

$$\cos^2\gamma = \frac{r_2^2}{r_2^2 + r_3^2}$$

$$\sin^2\gamma = \frac{r_3^2}{r_2^2 + r_3^2}$$

$$\sin^2\theta = \frac{r_2^2 + r_3^3}{r^2}$$

and equation (56) can be written as

$$\frac{r_1^2 r_j r_l}{r^4}\int U'U''R_{jl}\,d\vec{\eta} = \cos^4\theta \int U'U''R_{11}^+\,d\vec{\eta}$$

$$+ \cos^2\theta\,\sin^2\theta\,\cos^2\gamma \int U'U''R_{22}^+\,d\vec{\eta} + \cos^2\theta\,\sin^2\theta\,\sin^2\gamma \int U'U''R_{33}^+\,d\vec{\eta}$$

Using equation (45) and recalling that R_{ij}^+ and R_{ij}^- are even and odd functions of τ respectively, we find from equations (54) and (56) that

$$\frac{r_i r_j r_k r_l}{r^4}\int R_{ik}R_{jl}\,d\vec{\eta} = \cos^4\theta \int R_{11}^2\,d\vec{\eta} + 2\,\sin^2\theta\,\cos^2\theta \int \left(2R_{12}^2 + R_{11}R_{22}\right)d\vec{\eta}$$

$$+ \sin^4\theta \int \left(R_{22}^2\right)d\vec{\eta} + \textit{odd functions of } \tau$$

and

$$\frac{r_1 r_j r_l}{r^4}\int U'U''R_{jl}\,d\vec{\eta} = \cos^4\theta \int U'U''R_{11}\,d\vec{\eta} + \cos^2\theta\,\sin^2\theta\left(\cos^2\gamma \int U'U''R_{22}\,d\vec{\eta} + \sin^2\gamma \int U'U''R_{33}\,d\vec{\eta}\right) + \text{odd functions of } \tau$$

Recalling that odd functions of τ can make no contributions to the integral in equation (35), we can now write equation (35) as

$$I_\omega(\vec{x}|\vec{y}) = \frac{\rho_o}{16\pi^3 c_o^5 (1 - M_c\cos\theta)^4 r^2}\Bigg\{\left(S_{11}^{se}\cos^4\theta + 2S_{12}^{se}\cos^2\theta\,\sin^2\theta + S_{22}^{se}\sin^4\theta\right)$$

$$+ 2\left[S_1^{sh}\cos^4\theta + \cos^2\theta\,\sin^2\theta\left(S_2^{sh}\cos^2\gamma + S_3^{sh}\sin^2\gamma\right)\right]\Bigg\} \tag{57}$$

where we have put

$$S_{ij}^{se} \equiv \omega_r^4\,\mathcal{R}e \int_{-\infty}^{\infty} e^{-i\omega_r\tau}\,r_{ij}^{se}\,d\tau = \omega_r^4 \int_{-\infty}^{\infty} (\cos\omega_r\tau) r_{ij}^{se}\,d\tau \tag{58}$$

$$S_i^{sh} \equiv \omega_r^4 \, \mathcal{R}e \int_{-\infty}^{\infty} e^{-i\omega_r \tau} r_i^{sh} \, d\tau = \omega_r^4 \int_{-\infty}^{\infty} (\cos \omega_r \tau) r_i^{sh} \, d\tau \tag{59}$$

$$r_{ij}^{se} \equiv \int \left[R_{ii}R_{jj} + 2(1 - \delta_{ij})R_{ij}^2\right] d\vec{\eta} \quad \text{(no sum on i or j)} \tag{60}$$

$$r_i^{sh} \equiv \int U'U''R_{ii} \, d\vec{\eta} \qquad \text{(no sum on i)} \tag{61}$$

and where

$$\omega_r \equiv \omega(1 - M_c \cos \theta) \tag{62}$$

is the frequency in the reference frame moving with the eddies. Equation (57) expresses the spectral density of the far-field intensity emitted from a given point $\vec{y}$ in a convected axisymmetric turbulence.

ISOTROPIC TURBULENCE

In the special case of isotropic turbulence, the correlation tensor (37) becomes

$$R_{ij}(y, \vec{\eta}, \vec{\tau}) = A\eta_i\eta_j + B\delta_{ij} \tag{63}$$

where A and B are functions only of η and τ (see ref. 10). By introducing spherical coordinates in equation (60), it can be shown that

$$r_{ij}^{se} = r_{kl}^{se} \qquad \text{for } i,j,k,l = 1,2,3 \tag{64}$$

Upon inserting equation (63) into the continuity condition (38), we obtain

$$4A + \eta \frac{\partial A}{\partial \eta} + \frac{1}{\eta}\frac{\partial B}{\partial \eta} = 0$$

Hence (ref. 10), if we put

$$F \equiv B + \eta^2 A \tag{65}$$

this equation shows that

$$A = -\frac{1}{2}\frac{1}{\eta}\frac{\partial F}{\partial \eta} \tag{66}$$

Inserting these equations into equation (63) and then inserting the result into equation (61) with $i = 2$, we find that

$$r_2^{sh} \int = U\left(y_2 - \frac{\eta_2}{2}\right)U\left(y_2 + \frac{\eta_2}{2}\right)\left[F(\eta) + \frac{1}{2}\eta F'(\eta)\left(1 - \frac{\eta_2^2}{\eta^2}\right)\right]d\vec{\eta}$$

Introducing cylindrical coordinate λ, φ, η_2 with the axial direction along η_2, this becomes

$$r_2^{sh} = \pi \int_{-\infty}^{\infty} U\left(y_2 - \frac{\eta_2}{2}\right)U\left(y_2 + \frac{\eta_2}{2}\right)\int_0^{\infty}\left[2F(\eta) + \eta F'(\eta)\left(1 - \frac{\eta_2^2}{\eta^2}\right)\right]\lambda \, d\lambda \, d\eta_2$$

and since $\lambda^2 + \eta_2^2 = \eta^2$, we find upon changing variables that

$$\int_0^{\infty}\left[2F(\eta) + \eta F'(\eta)\left(1 - \frac{\eta_2^2}{\eta^2}\right)\right]\lambda \, d\lambda = \int_{|\eta_2|}^{\infty}\left[2F(\eta) + \eta F'(\eta)\left(1 - \frac{\eta_2^2}{\eta^2}\right)\right]\eta \, d\eta$$

$$= \int_{|\eta_2|}^{\infty} 2F(\eta)\eta \, d\eta + \left[F(\eta)\left(1 - \frac{\eta_2^2}{\eta^2}\right)\eta^2\right]_{|\eta_2|}^{\infty}$$

$$-2\int_{|\eta_2|}^{\infty} F(\eta)\eta \, d\eta = 0$$

Consequently,

$$r_2^{sh} = 0$$

It is also easy to see that

$$r_1^{sh} = r_3^{sh}$$

Notice that a similar analysis could be carried through in the moving frame if it were assumed that the moving axis correlation tensor (and not the fixed axis correlation tensor[4]) were isotropic.[5] However, this analysis would lead to the same result. Using these results together with equation (64) in equations (57), (58), and (59), we obtain

$$I_\omega(\vec{x}|\vec{y}) = \frac{\rho_o}{16\pi^3 c_o^5 \left(1 - M_c \cos\theta\right)^4 r^2}\left[S_{11}^{se} + 2S_1^{sh}\left(\cos^4\theta + \cos^2\theta \sin^2\theta \sin^2\gamma\right)\right] \tag{67}$$

AXISYMMETRIC FLOWS

Round jets constitute an important type of configuration for which the flow is axisymmetric. In this case equations (57) and (67) which are derived in a locally cartesian coordinate system at each point $\vec{y}$ should be averaged over the azimuthal angle γ (see ref. 4). Since the γ-dependence in equations (57) and (67) is explicitly shown, this averaging is easily accomplished and these equations become

For axisymmetric turbulence:

$$I_\omega(\vec{x}|\vec{y}) = \frac{\rho_o}{16\pi^3 c_o^5 \left(1 - M_c \cos\theta\right)^4 r^2}\left[\left(S_{11}^{se}\cos^4\theta + 2S_{12}^{se}\cos^2\theta \sin^2\theta + S_{22}^{se}\sin^4\theta\right)\right.$$

$$\left. + 2\left(S_1^{sh}\cos^4\theta + \frac{1}{2}S_T^{sh}\cos^2\theta\sin^2\theta\right)\right] \tag{68}$$

where

$$S_T^{sh} \equiv S_2^{sh} + S_3^{sh}$$

---

[4]They can not both be isotropic.

[5]This would actually be more appropriate, since the isotropic tensor must be symmetric in τ; and the data presented in refs. 10 and 18 show that the moving axis correlation tensor is much more symmetric in τ than the fixed axis correlation tensor.

For isotropic turbulence:

$$I_\omega(\vec{x}|\vec{y}) = \frac{\rho_o}{16\pi^3 c_o^5 \left(1 - M_c \cos\theta\right)^4 r^2}\left[S_{11}^{se} + 2S_1^{sh}\left(\cos^4\theta + \frac{1}{2}\cos^2\theta \sin^2\theta\right)\right] \tag{69}$$

OVERALL INTENSITY

The overall intensity $I(\vec{x}|\vec{y})$ at each field point is obtained by integrating the spectral density of the intensity over all frequencies. Thus

$$I(\vec{x}|\vec{y}) = \int_{-\infty}^{\infty} I_\omega(\vec{x}|\vec{y})d\omega$$

and equations (57) and (67) to (69) become, respectively,

General Flow

Axisymmetric turbulence:

$$I(\vec{x}|\vec{y}) = \frac{\rho_o}{8\pi^2 c_o^5 \left(1 - M_c \cos\theta\right)^5 r^2}\left\{\left(a_{11}\cos^4\theta + 2a_{12}\cos^2\theta\sin^2\theta + a_{22}\sin^4\theta\right)\right.$$

$$\left. + 2\left[b_1\cos^4\theta + \cos^2\theta\sin^2\theta\left(b_2\cos^2\gamma + b_3\sin^2\gamma\right)\right]\right\} \tag{70}$$

where

$$a_{ij} = \left.\frac{\partial^4}{\partial\tau^4} r_{ij}^{se}\right|_{\tau=0} \tag{71}$$

$$b_i = \left.\frac{\partial^4}{\partial\tau^4} r_i^{sh}\right|_{\tau=0} \tag{72}$$

Isotropic turbulence:

$$I(\vec{x}|\vec{y}) = \frac{\rho_o}{8\pi^2 c_o^5 \left(1 - M_c \cos\theta\right)^5 r^2} \left[a_{11} + 2b_1 (\cos^4\theta + \cos^2\theta \sin^2\theta \sin^2\gamma)\right] \tag{73}$$

Axisymmetric Flow

Axisymmetric turbulence:

$$I(\vec{x}|\vec{y}) = \frac{\rho_o}{8\pi^2 c_o^5 \left(1 - M_c \cos\theta\right)^5 r^2} \left\{\left(a_{11} \cos^4\theta + 2a_{12} \cos^2\theta \sin^2\theta + a_{22} \sin^4\theta\right) + 2\left[b_1 \cos^4\theta + \frac{1}{2}(b_2 + b_3) \cos^2\theta \sin^2\theta\right]\right\} \tag{74}$$

Isotropic turbulence:

$$I(\vec{x}|\vec{y}) = \frac{\rho_o}{8\pi^2 c_o^5 \left(1 - M_c \cos\theta\right)^5 r^2} \left[a_{11} + 2b_1\left(\cos^4\theta + \frac{1}{2} \cos^2\theta \sin^2\theta\right)\right] \tag{75}$$

SUMMARY OF APPROXIMATIONS

It is convenient to indicate explicitly the hierarchy of equations which is obtained by making the various approximations described previously to obtain the simplified equations starting from the essentially exact far-field equation (15). Thus, upon using equation (20) and collecting results we arrive at table I.

TABLE I. - HIERARCHY OF APPROXIMATIONS

$I_\omega(\vec{x}\vert\vec{y}) = \frac{\omega^4 \rho_o}{32\pi^3 c_o^5} \frac{r_i r_j r_k r_l}{r^6} \mathcal{R}e \iint e^{i\omega\left[(\vec{r}\cdot\vec{\eta}/c_o r)-\tau\right]} \mathscr{R}^{(1)}_{ijkl}(\vec{y},\vec{\eta},\tau)\,d\vec{\eta}\,d\tau$	Exact far-field equation (rewritten eq. (14))
↓ Small eddy Mach number	
$I_\omega(\vec{x}\vert\vec{y}) = \frac{\omega^4 \rho_o}{32\pi^3 c_o^5} \frac{r_i r_j r_k r_l}{r^6} \mathcal{R}e \int e^{-i\omega(1-M_c \cos\theta)\tau} \int \mathscr{R}_{ijkl}(\vec{y},\vec{\xi},\tau)\,d\vec{\xi}\,d\tau$	Eq. (24)
↓ Locally homogeneous turbulence	
$I_\omega(\vec{x}\vert\vec{y}) = \frac{\omega^4 \rho_o}{32\pi^3 c_o^5} \left[\frac{r_i r_j r_k r_l}{r^6} \mathcal{R}e \int e^{-i\omega(1-M_c \cos\theta)\tau} \int \tilde{R}_{ijkl}(\vec{y},\vec{\xi},\tau)\,d\vec{\xi}\,d\tau + \frac{4r_1^2 r_j r_l}{r^6} \mathcal{R}e \int e^{-i\omega(1-M_c \cos\theta)\tau} \int U'U''\tilde{R}_{jl}(\vec{y},\vec{\xi},\tau)\,d\vec{\xi}\,d\tau\right.$	Eq. (33)
↓ Normal joint probability distribution	
$I_\omega(\vec{x}\vert\vec{y}) = \frac{\omega^4 \rho_o}{16\pi^3 c_o^5} \left[\frac{r_i r_j r_k r_l}{r^6} \mathcal{R}e \int e^{-i\omega(1-M_c \cos\theta)\tau} \int R_{ik}R_{jl}\,d\vec{\eta}\,d\tau + \frac{2r_1^2 r_j r_l}{r^6} \mathcal{R}e \int e^{-i\omega(1-M_c \cos\theta)\tau} \int U'U''R_{jl}\,d\vec{\eta}\,d\tau\right]$	Eq. (36)
↓ Axisymmetric turbulence	
$I_\omega(\vec{x}\vert\vec{y}) = \frac{\rho_o}{16\pi^3 c_o^5 (1 - M_c \cos\theta)^4} \left\{\left(\cos^4\theta\, S_{11}^{se} + 2\cos^2\theta \sin^2\theta\, S_{12}^{se} + \sin^4\theta\, S_{22}^{se}\right) + 2\left[\cos^4\theta\, S_1^{sh} + \cos^2\theta \sin^2\theta \left(\cos^2\gamma\, S_2^{sh} + \sin^2\gamma\, S_3^{sh}\right)\right]\right\}$	Eq. (58)
↓ Isotropic turbulence	
$I_\omega(\vec{x}\vert\vec{y}) = \frac{\rho_o}{16\pi^3 c_o^5 (1 - M_c \cos\theta)^4} \left[S_{11}^{se} + 2S_1^{sh} (\cos^4\theta + \cos^2\theta \sin^2\theta \sin^2\gamma)\right]$	Eq. (67)

DISCUSSION OF RESULTS

Since $2\left(\cos^4\theta + \frac{1}{2}\cos^2\theta \sin^2\theta\right) = \cos^4\theta + \cos^2\theta$, equation (75) is exactly the result obtained by Ribner (ref. 4, eq. (31)) without including the convective amplification effect $\left(1 - M_c \cos\theta\right)^{-5}$. However, Ribner did include the convective amplification effect a posteriori (see ref. 4, eq. (34)) by using the higher order correction (in eddy convection Mach number) obtained by Ffowcs William in reference 9. It is important to notice that we obtained this equation without ever making the assumption that the correlation function factors into the product of a space-dependent part and a time-dependent part. Thus, the results obtained previously reduce to Ribner's results in the special case of isotropic turbulence when they are averaged over the azimuthal direction and this occurs without making the factorization assumption.

Now consider the case of an axisymmetric flow. Following Ribner we refer to the quantity in braces in equation (74) as the basic directivity pattern of the noise. Since this is the directivity pattern with convective amplification and refraction effects neglected, the first term

$$a_{11}\cos^4\theta + 2a_{12}\cos^2\theta\sin^2\theta + a_{22}\sin^4\theta$$

is the basic directivity pattern of the self-noise. The coefficient a_{11} of $\cos^4\theta$ consists of the self-correlation of the self-noise part of the longitudinal quadrupole τ_{11}. The coefficient a_{12} depends on the self-correlations of the two lateral quadrupoles τ_{12} and τ_{13} and the two cross correlations $(\overline{\tau'_{11}\tau''_{22}})_{self}$ and $(\overline{\tau'_{11}\tau''_{33}})_{self}$. Finally, the coefficient a_{22} of $\sin^4\theta$ depends on the average of the two self-correlations of the two longitudinal quadrupoles τ_{22} and τ_{33}, on the self-correlations of the lateral quadrupole τ_{23}, and on the cross correlation of the two longitudinal quadrupoles and the cross correlation of the longitudinal quadrupoles $\tau_{22}\tau_{33}$. The main contribution comes from the self-correlations of the longitudinal quadrupoles. Thus, the basic directivity of the self-noise term results from the directivity pattern of a large number of quadrupole correlations each of which has one of the directivities $\cos^4\theta$, $2\cos^2\theta\sin^2\theta$ or $\sin^4\theta$.

The second term in equation (74)

$$2b_1\cos^4\theta + (b_2 + b_3)\cos^2\theta\sin^2\theta$$

is the basic directivity pattern of the shear noise. The coefficient b_1 of $\cos^4\theta$ consists of the self-correlation of the shear noise part of the longitudinal quadrupole τ_{11}. The coefficient $(b_2 + b_3)$ of $\cos^2\theta\sin^2\theta$ consists of the self-correlation of the lateral quadrupoles τ_{12} and τ_{13}.

Thus the $\cos^4\theta$ term in equation (74) is basically due to a longitudinal quadrupole in the direction of flow and the $\sin^4\theta$ is basically due to longitudinal quadrupoles transverse to the flow. The $\cos^2\theta \sin^2\theta$ is due to lateral quadrupoles and cross correlations of longitudinal quadrupoles.

IMPLICATIONS FOR JET NOISE

As we have already indicated, the experiments of Jones (ref. 5) show that the correlation of the longitudinal quadrupoles in the direction of flow can be much larger than the correlation of the longitudinal quadrupoles transverse to the flow direction (see ref. 5, fig. 5). Thus, we might anticipate that the $\cos^4\theta$ term in the self-noise will be considerably larger than the $\sin^4\theta$ term. This would cause the self-noise in the mixing region to be beamed downstream. Since the spectrum of the shear noise peaks about one octave below the self-noise, this would at first seem to contradict the fact that the basic directivity of the high frequency sound tends to be nondirectional whereas the low frequency sound tends to be beamed downstream. However, this contradiction might be resolved when one considers the spectrum of the self-noise term. The small scale eddies will contribute mostly to the high frequency part of the spectrum of the self-noise. But these eddies tend to be much more isotropic than the large scale eddies which contribute mostly to the low frequency part of the spectrum. Thus the high frequency part of the self-noise spectrum will still be isotropic and only the low frequency part will be nonisotropic. This nonisotropy will result in a downstream beaming of the low frequency part of the self-noise so that both the low frequency part of the self-noise and the shear noise (which is all low frequency) will be beamed downstream. However, the high frequency part of the self-noise spectrum will be nondirectional. Hence, just as in the case of isotropic turbulence, we find that the high frequency sound is nondirectional while the low frequency sound is beamed downstream even though the self-noise term itself may be highly directional. The model of isotropic turbulence implies that this term is basically nondirectional (see eq. (75)).

We expect that the effects of nonisotropy of the turbulence can be significant in determining the directivity pattern of the narrow band noise as anticipated by Ribner (ref. 4, sec. 9).

For Ribner's model with isotropic turbulence (ref. 4) only the self-correlations of the longitudinal quadrupole τ_{11} and the lateral quadrupole τ_{13} contribute to the shear noise. In contrast, Lighthill (ref. 17) argued that the shear noise should be due almost entirely to the self-correlation of the τ_{12} lateral quadrupole which does not occur in Ribner's theory. However, the fuller pattern predicted by Ribner's result is in closer agreement with the experimental data (ref. 18). We see that, for axisymmetric turbulence, all three quadrupoles are present. This unifies these divergent viewpoints to some degree.

CONCLUDING REMARKS

A model has been developed to include the effects of nonisotropic turbulence in predicting jet noise. The turbulence model chosen is that of axisymmetric turbulence about the direction of flow. Thus, the statistical properties of the turbulence are independent of rotations about the direction of flow. The model reduces to Ribner's model in the special case when the turbulence is isotropic. However, Ribner's result is recovered without assuming, as Ribner did, that the turbulence correlation functions factor into a space-dependent and a time-dependent part. This shows that Ribner's model is actually more general than the assumptions he used to derive it would indicate. It is also shown that the only assumption required to assure that the intensity will be equal to the sum of a shear noise term and a self-noise term (with no cross coupling terms) is the relatively weak one that the turbulence be locally homogeneous.

It is indicated that the basic self-noise pattern may actually be more highly directional than previously anticipated and this can occur without contradicting the observed result that the low frequency noise is beamed downstream. The shear noise terms for axisymmetric turbulence contain both the quadrupoles of Ribner's model and those proposed by Lighthill which are missing from Ribner's model.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, May 26, 1972,
113-31.

APPENDIX - SYMBOLS

A	coefficient in axisymmetric tensor
$A^{\pm}$	even/odd part of A in τ
a_{ij}	fourth time derivative of correlation function
B	coefficient in axisymmetric tensor
$B^{\pm}$	even/odd part of B in τ
b_i	fourth time derivative of correlation function for shear noise
C	coefficient in axisymmetric tensor
$C^{\pm}$	even/odd part of C in τ
c_o	speed of sound
D	coefficient in axisymmetric tensor
$D^{\pm}$	even/odd part of C in τ
E	coefficient in axisymmetric tensor
$E^{\pm}$	even/odd part of D in τ
F	defined in eq. (65)
$G, G^{\pm}$	any coefficients of axisymmetric tensor (see eq. (41))
$I(\vec{x})$	intensity at point $\vec{x}$ of sound field
$I(\vec{x}\vert\vec{y})$	intensity at point $\vec{x}$ of sound emitted from point $\vec{y}$
$I_{\omega}(\vec{x})$	spectral density of $I(\vec{x})$
$I_{\omega}(\vec{x}\vert\vec{y})$	spectral density of $I(\vec{x}\vert\vec{y})$
l	correlation length of eddy
M_c	convection Mach number
M_e	eddy Mach number
Q	arbitrary function of $\vec{y}$, $\vec{z}$, and τ
q	arbitrary function of $\vec{z}-\vec{y}$ and τ
$\tilde{R}_{ijkl}, \tilde{R}_{i,jk}, \tilde{R}_{ij}$	moving frame correlation functions of turbulent velocities
R_{ij}	fixed frame correlation tensor of turbulent velocities
$\mathscr{R}_{ijkl}, \mathscr{R}^{+}_{ijkl}, \mathscr{R}^{c}_{ijkl}$	turbulence correlations in moving frame
$\mathscr{R}^{(1)}_{ijkl}, \mathscr{R}^{(2)}_{ijkl}$	turbulence correlations in fixed frame

$\vec{r}$	$\vec{x}-\vec{y}$, vector connecting source and observation points
r	magnitude of $\vec{r}$
r_i	component of $\vec{r}$
r_i^{sh}	shear noise correlations
r_{ij}^{se}	self-noise correlations
S_i^{sh}	proportional to Fourier transform of r_i^{sh}
S_{ij}^{se}	proportional to Fourier transform of r_{ij}^{se}
T_{ij}	Fourier transform of velocity correlations
t	time
U	mean velocity
$\vec{U}_c$	convection velocity
$\vec{u}$	turbulent velocity
u_i	component of $\vec{u}$
v_i	component of total velocity
$\vec{x}$	coordinate of observation point
x_i	component of $\vec{x}$
$\vec{y}$	average source coordinate; $y \equiv \left(y_1', \dfrac{y_2' + y''}{2}, \dfrac{y_3' + y_3''}{2}\right)$
$\vec{y}', \vec{y}''$	coordinates of source points
y_i	i^{th} component of $\vec{y}$
$\vec{z}$	coordinate of source point
z_i	component of $\vec{z}$
γ	azimuthal angle
Δ	Fourier transform of density fluctuation
$\vec{\xi}$	$\vec{\eta} - \vec{U}_c \tau$
$\vec{\eta}$	correlation distance $\vec{y}'' - \vec{y}'$
$\vec{\eta}^{(1)}$	$-\vec{\eta}$
η_i	component of $\vec{\eta}$
θ	angle between observation point and direction of mean flow
ρ	instantaneous density
ρ_o	mean density

τ time delay

ω frequency

Superscripts:

$'$ denotes quantity evaluated at point $\vec{y}'$ and time t

$''$ denotes quantity evaluated at point $\vec{y}'' = \vec{y}' + \vec{\eta}$ and time $t + \tau$

REFERENCES

1. Lighthill, M. J.: On Sound Generated Aerodynamically. I. General Theory. Proc. Roy. Soc. (London), Ser. A, vol. 211, no. 1107, Mar. 20, 1952, pp. 564-587.

2. Proudman, I.: The Generation of Noise by Isotropic Turbulence. Proc. Roy. Soc. (London), Ser. A, vol. 214, Aug. 7, 1952, pp. 119-132.

3. Lilley, G. M.: On the Noise from Air Jets. Rep. ARC-20376, Aeronautical Research Council, Gt. Britain, Sept. 8, 1958.

4. Ribner, H. S.: Quadrupole Correlations Governing the Pattern of Jet Noise. J. Fluid Mech., vol. 38, pt. 1, Aug. 14, 1969, pp. 1-24.

5. Jones, Ian S. F.: Fluctuating Turbulent Stresses in the Noise-Producing Region of a Jet. J. Fluid Mech., vol. 36, pt. 3, May 1, 1969, pp. 529-543.

6. Batchelor, G. K.: The Theory of Axisymmetric Turbulence. Proc. Roy. Soc. (London), Ser. A, vol. 186, no. 1007, Sept. 24, 1946, pp. 480-502.

7. Chandrasekhar, S.: The Theory of Axisymmetric Turbulence. Phil. Trans. Roy. Soc., Ser. A, vol. 242, no. 855, 1950, pp. 557-577.

8. Batchelor, George K.: The Theory of Homogeneous Turbulence. Cambridge University Press, 1960.

9. Ffowcs Williams, J. E.: The Noise from Turbulence Converted at High Speed. Phil. Trans. Roy. Soc., Ser. A, vol. 255, no. 1061, Apr. 18, 1963, pp. 469-503.

10. Davies, P. O. A. L.; Fisher, M. J.; and Barratt, M. J.: The Characteristics of the Turbulence in the Mixing Region of a Round Jet. J. Fluid Mech., vol. 15, pt. 3, Mar. 1963, pp. 337-367.

11. Lighthill, M. J.: Jet Noise. AIAA J., vol. 1, no. 7, July 1963, pp. 1507-1517.

12. Townsend, A. A.: The Structure of Turbulent Shear Flow. Cambridge University Press, 1956.

13. Ribner, H. S.: The Generation of Sound by Turbulent Jets. Advances in Applied Mechanics. Vol. 8. H. L. Dryden and Theodore von Kármán, eds., Academic Press, 1964, pp. 103-182.

14. Hinze, J. O.: Turbulence - An Introduction to its Mechanism and Theory. McGraw-Hill Book Co., Inc., 1959.

15. Batchelor, G. K.; and Proudman, I.: The Large-Scale Structure of Homogeneous Turbulence. Phil. Trans. Roy. Soc., Ser. A, vol. 248, no. 949, Jan. 5, 1956, pp. 369-405.

16. Beran, Mark J.: Statistical Continuum Theories. Interscience Publishers, 1968.

17. Lighthill, M. J.: On Sound Generated Aerodynamically. II. Turbulence as a Source of Sound. Proc. Roy. Soc. (London), Ser. A, vol. 222, no. 1148, Feb. 23, 1954, pp. 1-32.

18. Chu, Wing T.: Turbulence Measurements Relevant to Jet Noise. Rep. UTIAS-119, University of Toronto (AD-645322), Nov. 1966.

NASA-Langley, 1972 —— 23 E-6844

NATIONAL AERONAUTICS AND SPACE ADMISTRATION
WASHINGTON, D.C. 20546

OFFICIAL BUSINESS
PENALTY FOR PRIVATE USE $300

FIRST CLASS MAIL

POSTAGE AND FEES PAID
NATIONAL AERONAUTICS AND
SPACE ADMINISTRATION

U.S.MAIL

NASA 451

POSTMASTER: If Undeliverable (Section 158 Postal Manual) Do Not Return

---

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION OFFICE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Washington, D.C. 20546